Foldy-Lax Model for the Scattering Problem in Electromagnetism

Justine Labat, Victor Péron, Sébastien Tordeux

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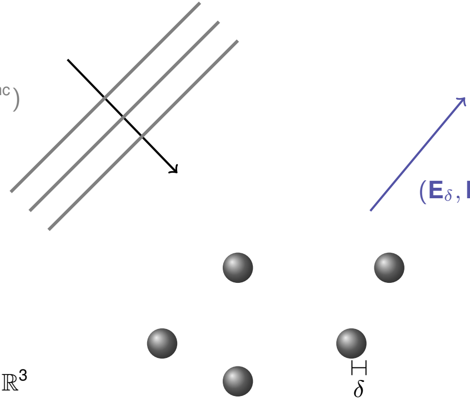
3D-Multiple Scattering Problem by Small Spheres

Incident field $(\mathbf{E}^{\mathrm{inc}}, \mathbf{H}^{\mathrm{inc}})$
with wavelength λ

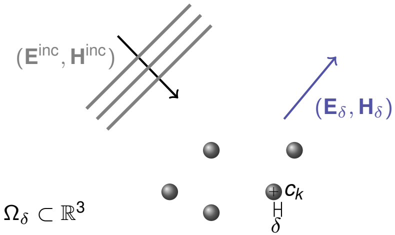


$(\mathbf{E}_\delta, \mathbf{H}_\delta)$ Scattered field

Exterior domain $\Omega_\delta \subset \mathbb{R}^3$

Asymptotic assumption : $\delta \ll \lambda$

Model Problem



- Time-harmonic domain
- Homogeneous & isotropic medium Ω_δ
- Perfect conductors $\mathcal{B}(c_k, \delta)$, $_{k=1...N_{obs}}$

Time-harmonic Maxwell equations

$$\begin{cases} \textbf{curl}\, \mathbf{E}_\delta - i\kappa\, \mathbf{H}_\delta = 0 & \text{in } \Omega_\delta \\ \textbf{curl}\, \mathbf{H}_\delta + i\kappa\, \mathbf{E}_\delta = 0 & \text{in } \Omega_\delta \end{cases}$$

with $\kappa^2 = \omega^2 \mu(\varepsilon + \frac{i\sigma}{\omega})$, $\Im(\kappa) \geq 0$

Boundary condition

$$\mathbf{n} \times \mathbf{E}_\delta = -\mathbf{n} \times \mathbf{E}^{inc} \qquad \text{on } \partial\Omega_\delta$$

Silver-Müller radiation condition

$$r\left(\mathbf{H}_\delta \times \widehat{x} - \mathbf{E}_\delta\right) \underset{r\to\infty}{\longrightarrow} 0 \quad \text{unif. in } \widehat{x} = \frac{\mathbf{x}}{r}$$

Mathematical well-posedness [Phillips (1973)]

For any $\mathbf{E}^{inc} \in \mathbf{H}_{loc}(\textbf{curl}\,, \Omega_\delta)$ there exist a unique solution $(\mathbf{E}_\delta, \mathbf{H}_\delta) \in \mathbf{H}_{loc}(\textbf{curl}\,, \Omega_\delta)^2$ to the exterior Maxwell problem.

Approaches and Objectives

Asymptotic models → Born approximation

Asymptotic models → Foldy-Lax approximation

H. Ammari, M.S. Vogelius, D. Volkov (2001)
Asymptotic formulas for perturbations in the electromagnetic fields due to the presence of inhomogeneities of small diameter II. The full Maxwell equations

D.P. Challa, G. Hu, M. Sini (2013)
Multiple scattering of electromagnetic waves by finitely many point-like obstacles

D.V. Korikov, B.A. Plamenevskii (2017)
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Numerical solution → High-order spectral method

M. Ganesh, S.C. Hawkins (2009)
A high-order algorithm for multiple electromagnetic scattering in three dimensions

H. Barucq, J. Chabassier, H. Pham, S. Tordeux (2017)
Numerical robustness of single-layer method with Fourier basis for multiple obstacle acoustic scattering in homogeneous media

Motivations

- ✗ **High computational cost** with classical numerical methods
- ✗ Mesh **refinement**
- ✓ High-order **spectral method**

Numerical Solution

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Numerical Solution

$\downarrow$ Superposition principle

Born approximation

- ✗ **High computational cost**
  - ✗ volumical methods
  - ∼ boundary element methods
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Numerical Solution

$\downarrow$ Superposition principle

Born approximation $\xrightarrow{\text{Approximation}}$ **Asymptotic models**

Born approximation:

- ✗ **High computational cost**
  - ✗ volumical methods
  - ∼ boundary element methods
- ✓ High-order **spectral method**

Asymptotic models:

- ✗ Restricted to small obstacles
- ✓ Low computational cost
- ✓ Meshless method

Motivations

- ✗ **High computational cost** with classical numerical methods
- ✗ Mesh **refinement**
- ✓ High-order **spectral method**

- ✓ Interactions taken into account
- ✓ Low computational cost
- ✓ Meshless method

Numerical Solution

↓ Superposition principle

Born approximation

Foldy-Lax model

↑ Multiple scattering

Asymptotic models

Approximation →

←

- ✗ **High computational cost**
  - ✗ volumical methods
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Outline

1. Asymptotic expansions for the single electromagnetic scattering
   - First terms of the asymptotics
   - Numerical results

2. Multiple electromagnetic scattering by small spheres
   - Born approximation
   - Foldy-Lax approximation
   - Preliminary numerical results

3. Conclusions and perspectives

Outline

1. Asymptotic expansions for the single electromagnetic scattering
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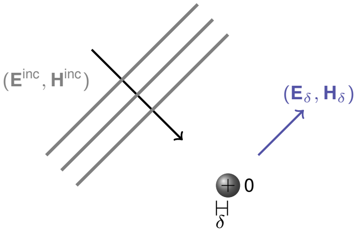
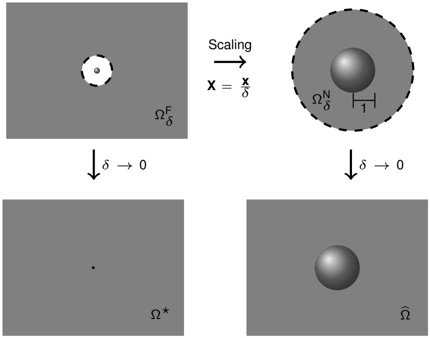
Approximation for Single Scattering

Method of matched asymptotic expansions

- Domain decomposition
- Local approximations
- Matching procedure





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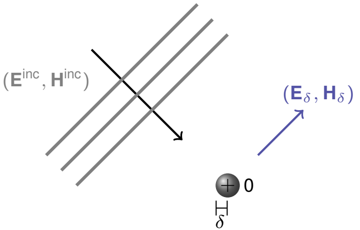

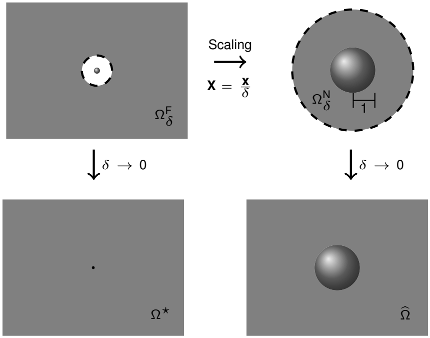

Asymptotic expansions

Far field expansion in $\Omega^\star = \mathbb{R}^3 \setminus \{0\}$

Near field expansion in $\widehat{\Omega} = \mathbb{R}^3 \setminus \overline{\mathcal{B}(0,1)}$

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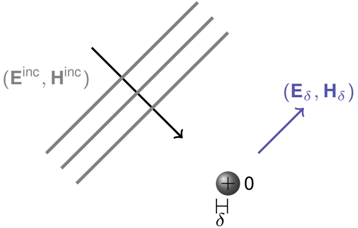
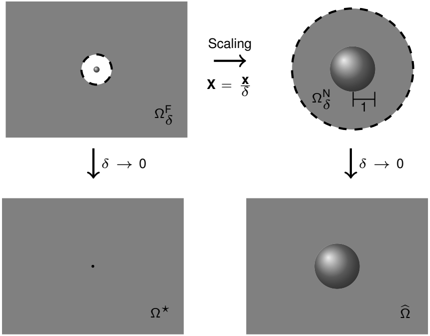




Asymptotic expansions

Far field expansion in $\Omega^\star = \mathbb{R}^3 \setminus \{0\}$

Near field expansion in $\widehat{\Omega} = \mathbb{R}^3 \setminus \overline{\mathcal{B}(0,1)}$

Near field $\Longrightarrow$ Far field

Near Field Expansion I

$$\mathbf{E}_\delta(\delta\mathbf{X}) = \widehat{\mathbf{E}}_0(\mathbf{X}) + \delta\widehat{\mathbf{E}}_1(\mathbf{X}) + \delta^2\widehat{\mathbf{E}}_2(\mathbf{X}) + \dots$$
$$\mathbf{H}_\delta(\delta\mathbf{X}) = \widehat{\mathbf{H}}_0(\mathbf{X}) + \delta\widehat{\mathbf{H}}_1(\mathbf{X}) + \delta^2\widehat{\mathbf{H}}_2(\mathbf{X}) + \dots$$

0-order (static **dipole**)

$$\widehat{\mathbf{E}}_0(\mathbf{X}) = \frac{1}{|\mathbf{X}|^3}\Big(3(\mathbf{e}_r \cdot \mathbf{E}^{\mathrm{inc}}(0))\mathbf{e}_r - \mathbf{E}^{\mathrm{inc}}(0)\Big)$$
$$\widehat{\mathbf{H}}_0(\mathbf{X}) = -\frac{1}{2|\mathbf{X}|^3}\Big(3(\mathbf{e}_r \cdot \mathbf{H}^{\mathrm{inc}}(0))\mathbf{e}_r - \mathbf{H}^{\mathrm{inc}}(0)\Big)$$

1-order (static **quadrupole** + static **dipole**)

$$\widehat{\mathbf{E}}_1(\mathbf{X}) = -\frac{1}{|\mathbf{X}|^4}\gamma_t\Big(\mathbb{J}^{\mathrm{sym}}_{\mathbf{E}^{\mathrm{inc}}}(0)\mathbf{e}_r\Big) + \frac{3}{2|\mathbf{X}|^4}(\mathbf{e}_r \cdot \mathbb{J}_{\mathbf{E}^{\mathrm{inc}}}(0)\mathbf{e}_r)\mathbf{e}_r - \frac{i\kappa}{2|\mathbf{X}|^2}\mathbf{H}^{\mathrm{inc}}(0) \times \mathbf{e}_r$$
$$\widehat{\mathbf{H}}_1(\mathbf{X}) = \frac{2}{3|\mathbf{X}|^4}\gamma_t\Big(\mathbb{J}^{\mathrm{sym}}_{\mathbf{H}^{\mathrm{inc}}}(0)\mathbf{e}_r\Big) - \frac{1}{|\mathbf{X}|^4}(\mathbf{e}_r \cdot \mathbb{J}_{\mathbf{H}^{\mathrm{inc}}}(0)\mathbf{e}_r)\mathbf{e}_r + \frac{i\kappa}{|\mathbf{X}|^2}\mathbf{E}^{\mathrm{inc}}(0) \times \mathbf{e}_r$$

with $\gamma_t\mathbf{u} = \mathbf{u} - (\mathbf{e}_r \cdot \mathbf{u})\mathbf{e}_r$ and $\mathbb{J}^{\mathrm{sym}} = \frac{1}{2}(\mathbb{J} + \mathbb{J}^\top)$ symmetric Jacobian

Near Field Expansion II

2-order (static **octupole** + static **quadrupole** + static **dipole**)

$$\widehat{\mathbf{E}}_2(\mathbf{X}) = \frac{1}{|\mathbf{X}|^5}\Big\{\frac{2}{3}(\mathbf{e}_r\cdot\mathbf{e}_r^\top \mathbb{H}_{\mathbf{E}^{\mathrm{inc}}}(0)\,\mathbf{e}_r)\mathbf{e}_r + \frac{2\kappa^2}{15}(\mathbf{e}_r\cdot\mathbf{E}^{\mathrm{inc}}(0))\mathbf{e}_r - \frac{1}{2}\gamma_t\big(\mathbf{e}_r^\top\,\mathbb{H}_{\mathbf{E}^{\mathrm{inc}}}(0)\,\mathbf{e}_r\big)$$
$$- \frac{i\kappa}{3}\mathbf{e}_r\times\mathbb{J}^{\mathrm{sym}}_{\mathbf{H}^{\mathrm{inc}}}(0)\,\mathbf{e}_r - \frac{\kappa^2}{5}\gamma_t\big(\mathbf{E}^{\mathrm{inc}}(0)\big)\Big\} + \frac{i\kappa}{3|\mathbf{X}|^3}\mathbf{e}_r\times\mathbb{J}^{\mathrm{sym}}_{\mathbf{H}^{\mathrm{inc}}}(0)\,\mathbf{e}_r$$
$$- \frac{3\kappa^2}{10|\mathbf{X}|^3}\Big\{\mathbf{E}^{\mathrm{inc}}(0) - 3(\mathbf{e}_r\cdot\mathbf{E}^{\mathrm{inc}}(0))\mathbf{e}_r\Big\} + \frac{\kappa^2}{2|\mathbf{X}|}\Big\{\mathbf{E}^{\mathrm{inc}}(0) + (\mathbf{e}_r\cdot\mathbf{E}^{\mathrm{inc}}(0))\mathbf{e}_r\Big\}$$

$$\widehat{\mathbf{H}}_2(\mathbf{X}) = \frac{1}{|\mathbf{X}|^5}\Big\{-\frac{1}{2}(\mathbf{e}_r\cdot\mathbf{e}_r^\top\,\mathbb{H}_{\mathbf{H}^{\mathrm{inc}}}(0)\,\mathbf{e}_r)\mathbf{e}_r - \frac{\kappa^2}{10}(\mathbf{e}_r\cdot\mathbf{H}^{\mathrm{inc}}(0))\mathbf{e}_r + \frac{3}{8}\gamma_t\big(\mathbf{e}_r^\top\,\mathbb{H}_{\mathbf{H}^{\mathrm{inc}}}(0)\,\mathbf{e}_r\big)$$
$$- \frac{i\kappa}{4}\mathbf{e}_r\times\mathbb{J}^{\mathrm{sym}}_{\mathbf{E}^{\mathrm{inc}}}(0)\mathbf{e}_r + \frac{3\kappa^2}{20}\gamma_t\big(\mathbf{H}^{\mathrm{inc}}(0)\big)\Big\} + \frac{i\kappa}{2|\mathbf{X}|}\mathbf{e}_r\times\mathbb{J}^{\mathrm{sym}}_{\mathbf{E}^{\mathrm{inc}}}(0)\,\mathbf{e}_r$$
$$+ \frac{3\kappa^2}{10|\mathbf{X}|^3}\Big\{3(\mathbf{e}_r\cdot\mathbf{H}^{\mathrm{inc}}(0))\mathbf{e}_r - \mathbf{H}^{\mathrm{inc}}(0)\Big\} - \frac{\kappa^2}{4|\mathbf{X}|}\Big\{\mathbf{H}^{\mathrm{inc}}(0) + (\mathbf{e}_r\cdot\mathbf{H}^{\mathrm{inc}}(0))\mathbf{e}_r\Big\}$$

Far Field Expansion I

$$\mathbf{E}_\delta(\mathbf{x}) = \delta^3 \widetilde{\mathbf{E}}_3(\mathbf{x}) + \delta^5 \widetilde{\mathbf{E}}_5(\mathbf{x}) + \dots \qquad \mathbf{H}_\delta(\mathbf{x}) = \delta^3 \widetilde{\mathbf{H}}_3(\mathbf{x}) + \delta^5 \widetilde{\mathbf{H}}_5(\mathbf{x}) + \dots$$

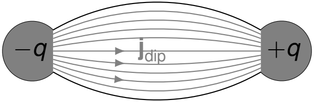
3-order (time-harmonic **dipole**)

$$\widetilde{\mathbf{E}}_3(\mathbf{x}) = -\frac{\kappa^3}{2} h_1^{(1)}(\kappa r)\mathbf{e}_r \times \mathbf{H}^{\mathrm{inc}}(0) - \kappa^3 \frac{\widetilde{h}_1^{(1)}(\kappa r)}{i\kappa r} \gamma_t \mathbf{E}^{\mathrm{inc}}(0) - 2\kappa^3 \frac{h_1^{(1)}(\kappa r)}{i\kappa r}(\mathbf{e}_r \cdot \mathbf{E}^{\mathrm{inc}}(0))\mathbf{e}_r$$

$$\widetilde{\mathbf{H}}_3(\mathbf{x}) = -\kappa^3 h_1^{(1)}(\kappa r)\mathbf{e}_r \times \mathbf{E}^{\mathrm{inc}}(0) + \frac{\kappa^3}{2} \frac{\widetilde{h}_1^{(1)}(\kappa r)}{i\kappa r} \gamma_t \mathbf{H}^{\mathrm{inc}}(0) + \kappa^3 \frac{h_1^{(1)}(\kappa r)}{i\kappa r}(\mathbf{e}_r \cdot \mathbf{H}^{\mathrm{inc}}(0))\mathbf{e}_r$$

with $h_n^{(1)}$ spherical Hankel of 1st kind and $\widetilde{h}_1^{(1)}(z) = h_1^{(1)}(z) + z h_1^{(1)'}(z)$

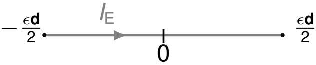
Time-Harmonic Electromagnetic Dipole



$\mathbf{j}_{\mathrm{dip}}$ electric current

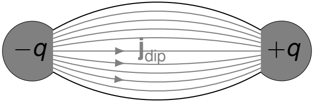
Ideal configuration

$\longrightarrow$



$\mathbf{d} \in \mathbb{R}^3$, ϵ small parameter

Time-Harmonic Electromagnetic Dipole



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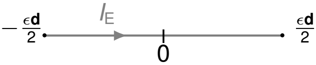

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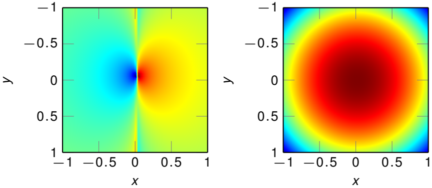
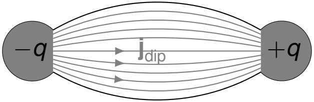


Time-harmonic electric potential V and magnetic potential $\mathbf{A}$ (x-component) in the xy-plane

$\epsilon \to 0$

- Ponctual charges with $q = \frac{1}{\epsilon}$
- Filiform current $(\mathbf{d}, I_{\mathrm{E}})$
- Charge conservation $I_{\mathrm{E}} = -i\omega q$

Time-Harmonic Electromagnetic Dipole



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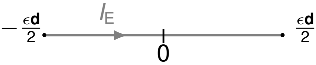

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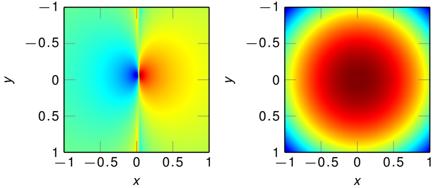

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- Filiform current $(\mathbf{d}, I_{\text{E}})$
- Charge conservation $I_{\text{E}} = -i\omega q$

$$\mathcal{E}_{\text{dip}}^{\text{elec}}[\mathbf{d}] = -\nabla V + i\kappa \mathbf{A}$$

$$\mathcal{H}_{\text{dip}}^{\text{elec}}[\mathbf{d}] = \mathbf{curl}\ \mathbf{A}$$

Magnetic dipole

$$\mathcal{H}_{\text{dip}}^{\text{mag}}[\mathbf{d}] = \mathcal{E}_{\text{dip}}^{\text{elec}}[\mathbf{d}]$$

$$\mathcal{E}_{\text{dip}}^{\text{mag}}[\mathbf{d}] = -\mathcal{H}_{\text{dip}}^{\text{elec}}[\mathbf{d}]$$

Far Field Expansion II

$$\mathbf{E}_\delta(\mathbf{x}) = \delta^3 \widetilde{\mathbf{E}}_3(\mathbf{x}) + \delta^5 \widetilde{\mathbf{E}}_5(\mathbf{x}) + \dots$$

3-order (time-harmonic **dipole**)

$$\widetilde{\mathbf{E}}_3 = \mathcal{E}_{\mathrm{dip}}^{\mathrm{elec}}[4\pi \mathbf{E}^{\mathrm{inc}}(0)] + \mathcal{E}_{\mathrm{dip}}^{\mathrm{mag}}[-2\pi \mathbf{H}^{\mathrm{inc}}(0)]$$

5-order (time-harmonic **dipole** + time-harmonic **quadrupole**)

$$\widetilde{\mathbf{E}}_5(\mathbf{x}) = \frac{3\kappa^2}{10} \mathcal{E}_{\mathrm{dip}}^{\mathrm{elec}}[4\pi \mathbf{E}^{\mathrm{inc}}(0)](\mathbf{x}) - \frac{3\kappa^2}{5} \mathcal{E}_{\mathrm{dip}}^{\mathrm{mag}}[-2\pi \mathbf{H}^{\mathrm{inc}}(0)](\mathbf{x})$$
$$- \frac{\kappa^4}{9} h_2^{(1)}(\kappa r)\mathbf{e}_r \times \mathbb{J}^{\mathrm{sym}}_{\mathbf{H}^{\mathrm{inc}}}(0)\mathbf{e}_r - \frac{\kappa^4}{6} \frac{\widetilde{h}_2^{(1)}(\kappa r)}{i\kappa r} \gamma_t \left(\mathbb{J}^{\mathrm{sym}}_{\mathbf{E}^{\mathrm{inc}}}(0)\mathbf{e}_r \right) - \frac{\kappa^4}{2} \frac{h_2^{(1)}(\kappa r)}{i\kappa r} (\mathbf{e}_r \cdot \mathbb{J}_{\mathbf{E}^{\mathrm{inc}}}(0)\mathbf{e}_r)\mathbf{e}_r$$

Collected dipolar model

$$\mathbf{E}_\delta^{\mathrm{Col}}(\mathbf{x}) = \left(\delta^3 + \frac{3\kappa^2\delta^5}{10}\right) \mathcal{E}_{\mathrm{dip}}^{\mathrm{elec}}[4\pi \mathbf{E}^{\mathrm{inc}}(0)](\mathbf{x}) + \left(\delta^3 - \frac{3\kappa^2\delta^5}{5}\right) \mathcal{E}_{\mathrm{dip}}^{\mathrm{mag}}[-2\pi \mathbf{H}^{\mathrm{inc}}(0)](\mathbf{x})$$

Far Field Expansion III

$$\mathbf{H}_\delta(\mathbf{x}) = \delta^3 \widetilde{\mathbf{H}}_3(\mathbf{x}) + \delta^5 \widetilde{\mathbf{H}}_5(\mathbf{x}) + \dots$$

3-order (time-harmonic **dipole**)

$$\widetilde{\mathbf{H}}_3 = \mathcal{H}^{\text{elec}}_{\text{dip}}[4\pi\mathbf{E}^{\text{inc}}(0)] + \mathcal{H}^{\text{mag}}_{\text{dip}}[-2\pi\mathbf{H}^{\text{inc}}(0)]$$

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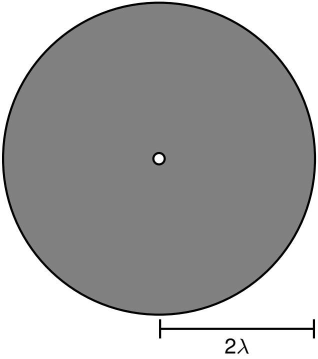
$$- \frac{\kappa^4}{6} h_2^{(1)}(\kappa r)\mathbf{e}_r \times \mathbb{J}^{\text{sym}}_{\mathbf{E}\text{inc}}(0)\mathbf{e}_r + \frac{\kappa^4}{9}\frac{\widetilde{h}_2^{(1)}(\kappa r)}{i\kappa r}\gamma_t\left(\mathbb{J}^{\text{sym}}_{\mathbf{H}\text{inc}}(0)\mathbf{e}_r\right) + \frac{\kappa^4}{3}\frac{h_2^{(1)}(\kappa r)}{i\kappa r}(\mathbf{e}_r \cdot \mathbb{J}_{\mathbf{H}\text{inc}}(0)\mathbf{e}_r)\mathbf{e}_r$$

Collected dipolar model

$$\mathbf{H}^{\text{Col}}_\delta(\mathbf{x}) = \left(\delta^3 + \frac{3\kappa^2\delta^5}{10}\right)\mathcal{H}^{\text{elec}}_{\text{dip}}[4\pi\mathbf{E}^{\text{inc}}(0)](\mathbf{x}) + \left(\delta^3 - \frac{3\kappa^2\delta^5}{5}\right)\mathcal{H}^{\text{mag}}_{\text{dip}}[-2\pi\mathbf{H}^{\text{inc}}(0)](\mathbf{x})$$

Simulation Description

3D Whole domain



Small parameter

$$\delta \in \left\{ \frac{1}{10^p},\ p = 0.5 : 0.1 : 4 \right\}$$

Physical parameters

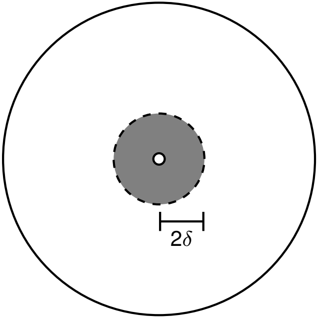
- $\varepsilon = \mu = 1.0, \quad \sigma = 0$
- $\lambda = 5.0,$
  $\omega = \kappa = \frac{2\pi}{5.0} \approx 1.26$

Incident plane wave

- $\mathbf{E}^{\text{inc}} = \vec{p}\exp(-i\vec{k}\cdot\mathbf{x})$
- $\mathbf{H}^{\text{inc}} = \frac{1}{i\kappa}\,\mathbf{curl}\;\mathbf{E}^{\text{inc}}$
- $\vec{p} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},\ \vec{k} = \begin{pmatrix} 0 \\ 0 \\ \kappa \end{pmatrix}$

Simulation Description

Near field domain



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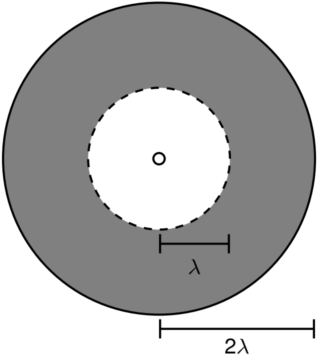
Simulation Description

Far field domain



Small parameter

$$\delta \in \left\{ \frac{1}{10^p},\ p = 0.5 : 0.1 : 4 \right\}$$

Physical parameters

- $\varepsilon = \mu = 1.0, \quad \sigma = 0$
- $\lambda = 5.0,$
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Incident plane wave

- $\mathbf{E}^{\text{inc}} = \vec{p} \exp(-i\vec{k} \cdot \mathbf{x})$
- $\mathbf{H}^{\text{inc}} = \frac{1}{i\kappa} \mathbf{curl}\, \mathbf{E}^{\text{inc}}$
- $\vec{p} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},\ \vec{k} = \begin{pmatrix} 0 \\ 0 \\ \kappa \end{pmatrix}$

Numerical Convergence I

Near field approximations

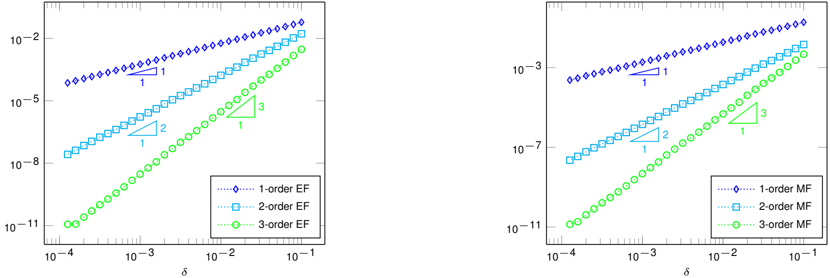
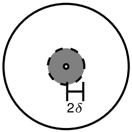


Figure: Relative $\mathbf{L}^2(\Omega_\delta^{2\delta})$-errors for electric near field (left) magnetic near field (right)



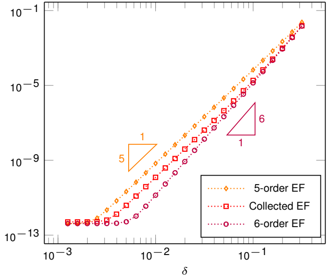
1-order EF : $\|\mathbf{E}_\delta - \widehat{\mathbf{E}}_0(\frac{\cdot}{\delta})\|_{\mathbf{L}^2(\Omega_\delta^{2\delta})} / \|\mathbf{E}_\delta\|_{\mathbf{L}^2(\Omega_\delta^{2\delta})}$

2-order EF : $\|\mathbf{E}_\delta - \widehat{\mathbf{E}}_0(\frac{\cdot}{\delta}) - \delta\widehat{\mathbf{E}}_1(\frac{\cdot}{\delta})\|_{\mathbf{L}^2(\Omega_\delta^{2\delta})} / \|\mathbf{E}_\delta\|_{\mathbf{L}^2(\Omega_\delta^{2\delta})}$

3-order EF : $\|\mathbf{E}_\delta - \widehat{\mathbf{E}}_0(\frac{\cdot}{\delta}) - \delta\widehat{\mathbf{E}}_1(\frac{\cdot}{\delta}) - \delta^2\widehat{\mathbf{E}}_2(\frac{\cdot}{\delta})\|_{\mathbf{L}^2(\Omega_\delta^{2\delta})} / \|\mathbf{E}_\delta\|_{\mathbf{L}^2(\Omega_\delta^{2\delta})}$

Numerical Convergence II

Far field approximations



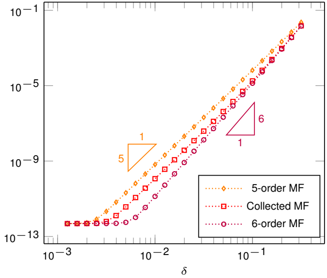
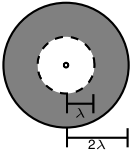


Figure: Absolute $\mathbf{L}^2(\Omega_\lambda^{2\lambda})$-errors for electric far field (left) magnetic far field (right)



5-order EF : $\|\mathbf{E}_\delta - \delta^3 \widetilde{\mathbf{E}}_3\|_{\mathbf{L}^2(\Omega_\lambda^{2\lambda})}$

6-order EF : $\|\mathbf{E}_\delta - \delta^3 \widetilde{\mathbf{E}}_3 - \delta^5 \widetilde{\mathbf{E}}_5\|_{\mathbf{L}^2(\Omega_\lambda^{2\lambda})}$

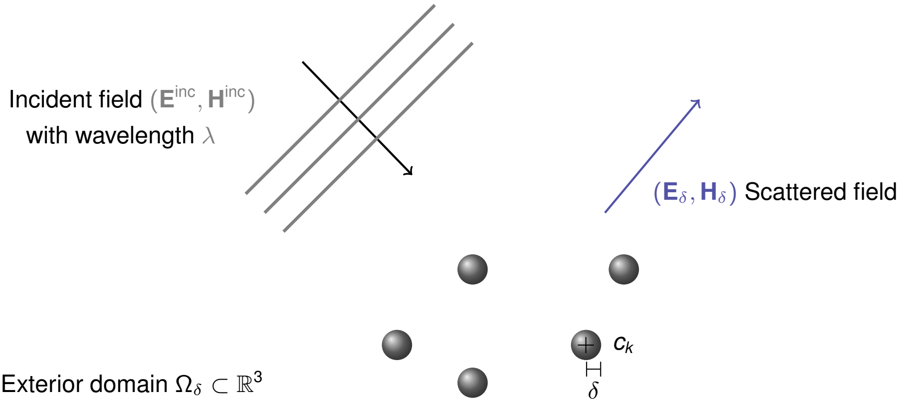
Outline

1. Asymptotic expansions for the single electromagnetic scattering
   - First terms of the asymptotics
   - Numerical results

2. Multiple electromagnetic scattering by small spheres
   - Born approximation
   - Foldy-Lax approximation
   - Preliminary numerical results

3. Conclusions and perspectives

3D-Multiple Scattering Problem by Small Spheres



$$\boxed{\Omega_\delta = \mathbb{R}^3 \setminus \bigcup_{k=1}^{N_{\text{obs}}} \overline{\mathcal{B}(c_k, \delta)}}$$

Born Approximation

$$\mathbf{E}_\delta^{\text{Born}}(\mathbf{x}) = \sum_{k=1}^{N_{\text{obs}}} \mathbf{E}_{\delta,k}(\mathbf{x}) \qquad \mathbf{H}_\delta^{\text{Born}}(\mathbf{x}) = \sum_{k=1}^{N_{\text{obs}}} \mathbf{H}_{\delta,k}(\mathbf{x})$$

- Each obstacle is modeled as a **dipolar source** around c_k

$$\mathbf{E}_{\delta,k}(\mathbf{x}) = \mathcal{E}_{\text{dip}}^{\text{elec}}[\mathbf{d}_{\delta,k}^{\text{E}}](\mathbf{x} - c_k) + \mathcal{E}_{\text{dip}}^{\text{mag}}[\mathbf{d}_{\delta,k}^{\text{H}}](\mathbf{x} - c_k)$$
$$\mathbf{H}_{\delta,k}(\mathbf{x}) = \mathcal{H}_{\text{dip}}^{\text{elec}}[\mathbf{d}_{\delta,k}^{\text{E}}](\mathbf{x} - c_k) + \mathcal{H}_{\text{dip}}^{\text{mag}}[\mathbf{d}_{\delta,k}^{\text{H}}](\mathbf{x} - c_k)$$

- The directions $\mathbf{d}_{\delta,k}^{\text{E}}$ and $\mathbf{d}_{\delta,k}^{\text{H}}$ depend on the **nature** of the obstacles

Born Approximation

$$\mathbf{E}_\delta^{\mathrm{Born}}(\mathbf{x}) = \sum_{k=1}^{N_{\mathrm{obs}}} \mathbf{E}_{\delta,k}(\mathbf{x}) \qquad \mathbf{H}_\delta^{\mathrm{Born}}(\mathbf{x}) = \sum_{k=1}^{N_{\mathrm{obs}}} \mathbf{H}_{\delta,k}(\mathbf{x})$$

- Each obstacle is modeled as a **dipolar source** around c_k

$$\mathbf{E}_{\delta,k}(\mathbf{x}) = \mathcal{E}_{\mathrm{dip}}^{\mathrm{elec}}[\mathbf{d}_{\delta,k}^{\mathrm{E}}](\mathbf{x}-c_k) + \mathcal{E}_{\mathrm{dip}}^{\mathrm{mag}}[\mathbf{d}_{\delta,k}^{\mathrm{H}}](\mathbf{x}-c_k)$$

$$\mathbf{H}_{\delta,k}(\mathbf{x}) = \mathcal{H}_{\mathrm{dip}}^{\mathrm{elec}}[\mathbf{d}_{\delta,k}^{\mathrm{E}}](\mathbf{x}-c_k) + \mathcal{H}_{\mathrm{dip}}^{\mathrm{mag}}[\mathbf{d}_{\delta,k}^{\mathrm{H}}](\mathbf{x}-c_k)$$

- The directions $\mathbf{d}_{\delta,k}^{\mathrm{E}}$ and $\mathbf{d}_{\delta,k}^{\mathrm{H}}$ depend on the **nature** of the obstacles

Case of perfectly conducting spheres

- 3-order approximation

$$\mathbf{d}_{\delta,k}^{\mathrm{E}} = 4\pi\delta^3\mathbf{E}^{\mathrm{inc}}(c_k) \quad \mathbf{d}_{\delta,k}^{\mathrm{H}} = -2\pi\delta^3\mathbf{H}^{\mathrm{inc}}(c_k)$$

- Corrected dipolar approximation

$$\mathbf{d}_{\delta,k}^{\mathrm{E}} = 4\pi\delta^3\left(1+\frac{3(\kappa\delta)^2}{10}\right)\mathbf{E}^{\mathrm{inc}}(c_k)$$

$$\mathbf{d}_{\delta,k}^{\mathrm{H}} = -2\pi\delta^3\left(1-\frac{3(\kappa\delta)^2}{5}\right)\mathbf{H}^{\mathrm{inc}}(c_k)$$

Important restrictions

- Very small obstacles
- Small number of obstacles
- No interaction taken into account

Foldy-Lax Approximation

$$\mathbf{E}_\delta^{\text{Foldy}}(\mathbf{x}) = \sum_{k=1}^{N_{\text{obs}}} \mathbf{E}_{\delta,k}(\mathbf{x}) \qquad \mathbf{H}_\delta^{\text{Foldy}}(\mathbf{x}) = \sum_{k=1}^{N_{\text{obs}}} \mathbf{H}_{\delta,k}(\mathbf{x})$$

- Each obstacle is modeled as a **dipolar source** around c_k

$$\mathbf{E}_{\delta,k}(\mathbf{x}) = \mathcal{E}_{\text{dip}}^{\text{elec}}[\mathbf{d}_{\delta,k}^{\text{E}}](\mathbf{x} - c_k) + \mathcal{E}_{\text{dip}}^{\text{mag}}[\mathbf{d}_{\delta,k}^{\text{H}}](\mathbf{x} - c_k)$$

$$\mathbf{H}_{\delta,k}(\mathbf{x}) = \mathcal{H}_{\text{dip}}^{\text{elec}}[\mathbf{d}_{\delta,k}^{\text{E}}](\mathbf{x} - c_k) + \mathcal{H}_{\text{dip}}^{\text{mag}}[\mathbf{d}_{\delta,k}^{\text{H}}](\mathbf{x} - c_k)$$

Case of perfectly conducting spheres

- 3-order approximation

$$\mathbf{d}_{\delta,k}^{\text{E}} = 4\pi\delta^3 \left(\mathbf{E}^{\text{inc}}(c_k) + \sum_{\substack{\ell=1 \\ \ell \neq k}}^{N_{\text{obs}}} \mathbf{E}_{\delta,\ell}(c_k) \right) \qquad \mathbf{d}_{\delta,k}^{\text{H}} = -2\pi\delta^3 \left(\mathbf{H}^{\text{inc}}(c_k) + \sum_{\substack{\ell=1 \\ \ell \neq k}}^{N_{\text{obs}}} \mathbf{H}_{\delta,\ell}(c_k) \right)$$

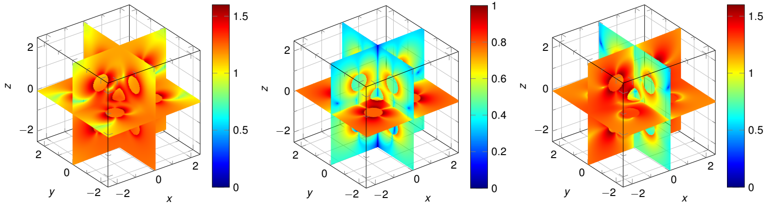
- Corrected dipolar approximation

$$\mathbf{d}_{\delta,k}^{\text{E}} = 4\pi\delta^3 \left(1 + \frac{3(\kappa\delta)^2}{10}\right) \left(\mathbf{E}^{\text{inc}}(c_k) + \sum_{\substack{\ell=1 \\ \ell \neq k}}^{N_{\text{obs}}} \mathbf{E}_{\delta,\ell}(c_k) \right) \qquad \mathbf{d}_{\delta,k}^{\text{H}} = -2\pi\delta^3 \left(1 - \frac{3(\kappa\delta)^2}{5}\right) \left(\mathbf{H}^{\text{inc}}(c_k) + \sum_{\substack{\ell=1 \\ \ell \neq k}}^{N_{\text{obs}}} \mathbf{H}_{\delta,\ell}(c_k) \right)$$

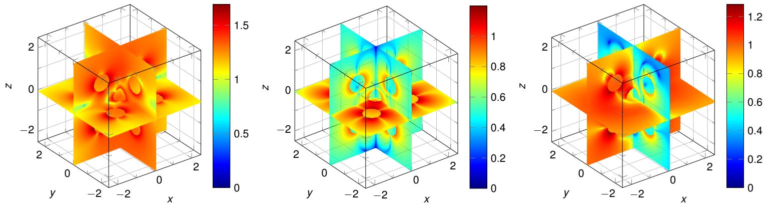
First Numerical Results

13 obstacles ; $d_{jk} \approx 1.0$; $\delta = 0.4$

Born approximation



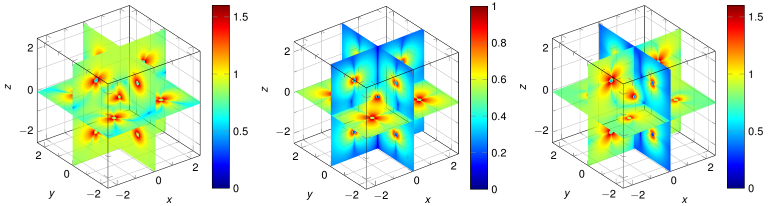
Foldy-Lax approximation



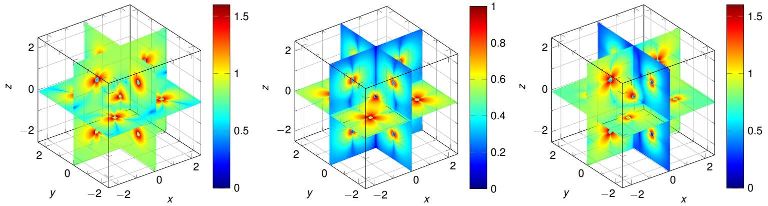
First Numerical Results

13 obstacles ; $d_{jk} \approx 1.25$; $\delta = 0.1$

Born approximation



Foldy-Lax approximation



Outline

Conclusions and Perspectives

Conclusions and on-going works

- Single scattering
  - ✓ Derivation of first terms of asymptotics for one sphere
  - ✓ Numerical validation

J. Labat, V. Péron, S. Tordeux (Inria Research Report n°9169, 2018)
Asymptotic Modeling of the Electromagnetic Scattering by Small Spheres Perfectly Conducting.

- Multiple scattering
  - ✓ Born approximation
  - ✓ Foldy-Lax approximation
  - ∼ High-order spectral method
  - ∼ Numerical validation of Born and Foldy-Lax models

Perspectives

- Single scattering
  - ✗ Justification of matched asymptotic expansions
  - ✗ Extension to obstacles of arbitrary shape
- Multiple scattering
  - ✗ Justification of Foldy-Lax approximation
  - ✗ Justification of the high-order spectral method